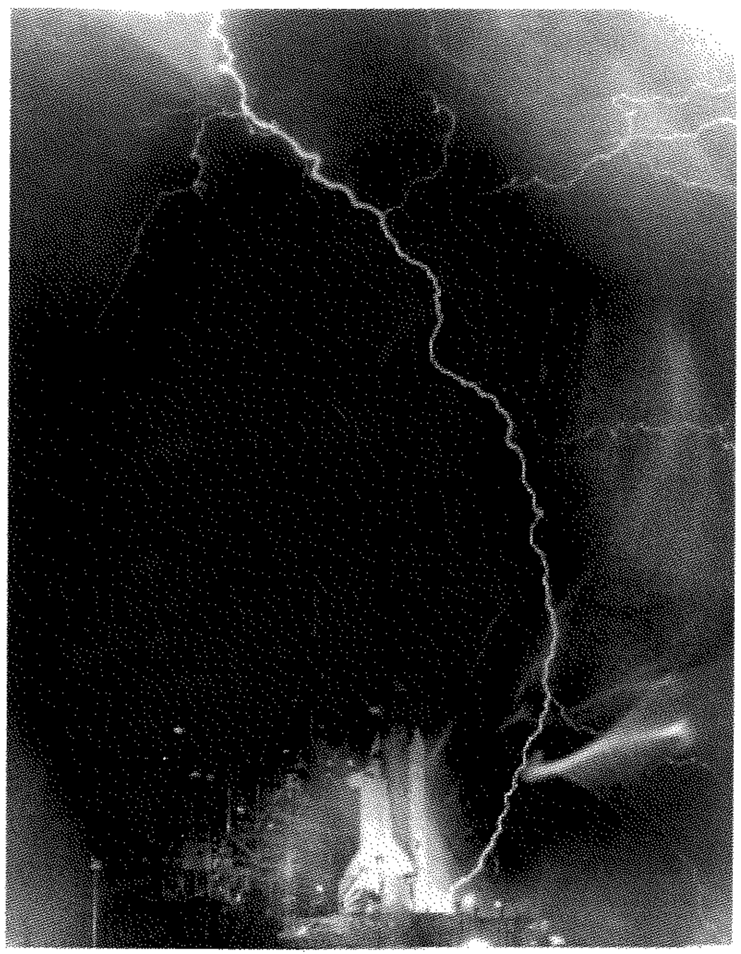
Six Ideas That Shaped Physics

Unit E: Electric and Magnetic Fields Are Unified



Thomas A. Moore

Third Edition (Draft)

E15 Introduction to Waves

- Static Electric Fields
- Controlling Currents
- Static Magnetic Fields
- Field Equations
- Dynamic Fields
  - Maxwell's Equations
  - Induction
  - Introduction to Waves
  - Electromagnetic Waves

Chapter Overview

Introduction

Maxwell's crowning achievement was his discovery that his equations allowed for the possibility of waves moving through an electromagnetic field. In chapter E16 we will explore the nature of these wave-like solutions of Maxwell's equations. When we talk about "electromagnetic waves," though, we are really constructing an analogy to mechanical waves (such as water waves) with which we have more experience. The purpose of *this* chapter is, therefore, to discuss how we can describe such waves physically and mathematically, so that we can better appreciate the analogy.

Section E15.1: What is a Wave?

In general, a **wave** is *a disturbance that moves through a medium while the medium remains essentially at rest*. Examples include **water waves, sound waves, tension waves** on a vibrating string or spring, **seismic waves**, and "the wave" at a stadium.

In this chapter, we will focus primarily on **mechanical waves**, where the disturbance involves some kind of physical displacement of the medium. Almost all such waves can be classified as being either **transverse** or **longitudinal**, which involve displacements of the medium that are *perpendicular* to or *parallel to* the wave motion respectively. Such mechanical waves can carry energy from place to place.

Section E15.2: A Sinusoidal Wave

Fourier's theorem states that a wave of any shape can be treated as a superposition of **sinusoidal waves**. Therefore, if we fully understand how sinusoidal waves behave in a given situation, we essentially understand how *any* wave would behave.

In this unit, we will consider only **one-dimensional waves**, waves whose disturbance function $f(t,x)$ depends on time and only one spatial coordinate. The equations describing a one-dimensional sinusoidal wave and is associated quantities are

$$f(t,x) = A\sin(kx - \omega t) \qquad \text{(E15.7)}$$

$$\text{where} \quad k \equiv \frac{2\pi}{\lambda}, \quad \omega \equiv \frac{2\pi}{T}, \quad f = \frac{\omega}{2\pi} = \frac{1}{T} \qquad \text{(E15.8)}$$

Purpose: These equations describe an idealized one-dimensional sinusoidal wave that varies with time t and position x along the x axis.

Symbols: $f(t,x)$ quantifies the "disturbance" the wave represents at point x at time t, A is the wave's **amplitude**, k (not the Coulomb k!) its **wavenumber**, λ its **wavelength**, ω its **angular frequency**, T its **period**, and f its **frequency** (don't confuse this with the disturbance function).

Limitations: This is an idealization of a real wave.

The value of the disturbance oscillates from $+A$ to $-A$, where A is the wave's *amplitude*. The ***wavenumber*** k expresses (in radians per meter) how rapidly the wave oscillates with increasing position at a given instant. It is related to the ***wavelength*** λ, which specifies the distance between wave crests at a given instant. The ***angular frequency*** ω specifies (in radians per second) how rapidly the wave oscillates with increasing time at a given position. It is related to the ***frequency*** f of the wave (the number of complete oscillations per unit time at a given position), and the wave's ***period*** T (which is the time required for a complete oscillation at a given position).

Section E15.3: The Phase Velocity of a Wave

Another important feature of the sinusoidal wave $f(t,x) = A\sin(kx - \omega t)$ is that its shape moves to the right as time passes. Many of the waves we encounter in nature are **traveling waves** of this type. In this section, we see that a given crest of a sinusoidal wave moves in the $+x$ direction with a **phase speed** v of

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f \tag{E15.12}$$

Purpose: This equation describes how we can calculate a sinusoidal wave's phase speed v from information about its angular velocity ω, its wavenumber k, its wavelength λ, its period T and/or its frequency f.

Limitations: This expression applies only to sinusoidal traveling waves.

(A wave's **phase velocity** $\vec{v}$ specifies the direction as well as the rate of the motion.)

Section E15.4: The Wave Equation

One of the most important equations in physics is the **wave equation**:

$$0 = b\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} \tag{E15.14}$$

$$\text{where } v = \frac{1}{\sqrt{b}} \tag{E15.19}$$

Purpose: If this equation (where b is a constant independent of t and x) accurately describes the behavior of a disturbance $f(t,x)$ in a medium, that disturbance will travel through the medium as a traveling wave moving in the $\pm x$ direction with phase speed v.

Limitations: This equation applies only to cases where the disturbance depends only on one spatial coordinate x.

Note: Remember that when we evaluate the partial derivative of $f(t,x)$ with respect to one of the variables t or x, we treat the other variable as if it were a constant.

A medium where disturbances obey this wave equation has a number of nice properties: (1) the medium supports sinusoidal traveling waves, (2) it obeys the principle of superposition, and (3) waves of arbitrary shape preserve their shape as they move.

This section explores examples of several kinds of wave-carrying media and show that they do obey the wave equation. We will also see why the wave equation implies the existence of *traveling* waves.

We will find this equation every useful in chapter E16.

E15.1 What Is a Wave?

Drop a pebble in a still pond; the splash of the pebble creates a series of concentric ripples that move out from the disturbance at a sedate and constant pace. When these ripples arrive at the location of a small object floating in the pond (such as a leaf or small stick) some distance away, they cause the object to bob up and down. The fact that the object bobs up and down instead of being swept in the direction of the wave's motion indicates that the pond water that carries the wave does not substantially move along with the wave. The waves move *through* the medium of the water: while the water itself is disturbed by the passing wave and moves slightly in response to it, there is no net displacement of the water in the direction of the wave.

There are many kinds of waves in the natural world

A **wave** in general can be described as being *a disturbance that moves through a medium while the medium remains basically at rest*, at least compared to the velocity of the wave. Examples of such waves in nature are abundant: **water waves** (from tiny ripples to tsunamis), **sound waves** (ranging from tiny whispers to explosion shock waves), **tension waves** on a vibrating string or spring, **seismic waves** that radiate through the Earth's crust from an earthquake, and so on. Figure E15.1 shows some physical waves.

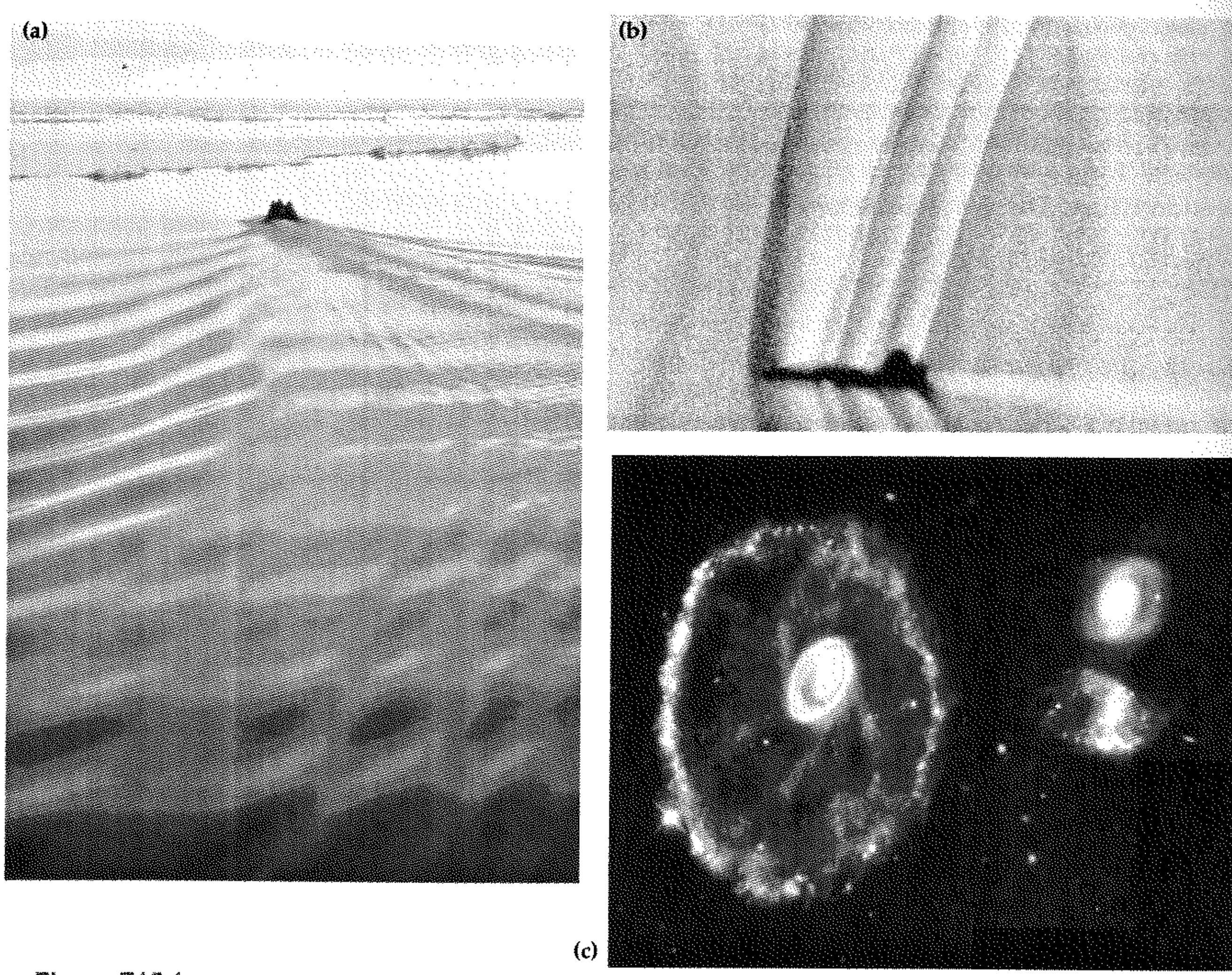


Figure E15.1
Various examples of waves. (a) Water waves from a boat moving through still water. (b) A Schlieren photograph of the shock waves in the air surrounding a supersonic jet. (c) A Hubble photograph of the Cartweel Galaxy. A head-on collision with another galaxy has caused a circular shock wave to move radially outward through the galaxy's gas. The wave compresses the gas, causing a burst of star formation just behind the wave's leading edge

The list given above by no means exhausts the kinds of waves that occur in nature. A crowd doing "The wave" in a stadium provides a good example of a disturbance that moves through a medium (in this case, the human beings involved the wave) without a net motion of the medium in the direction of the wave's motion. If you observe a traffic jam from a helicopter, you can sometimes see waves of disturbance radiate through obstructed traffic at speeds much higher than any of the individual cars are moving. Recently, astrophysicists have discovered that star formation in galaxies often moves in waves away from some disturbance in the galaxy's structure (say as the result of a collision with another galaxy). The growth of cells in a Petri dish can sometimes proceed in waves. The list goes on and on.

Indeed, waves occur so commonly in the physical world and in such a wide variety of contexts that a general study of wave behavior is an indispensable part of a physicist's education. Studying wave behavior in this course would be worthwhile in and of itself even if we weren't using it here as a stepping stone for the study of electromagnetic waves.

While waves of star formation or biological growth are definitely "disturbances in a medium", we will focus in the next few sections on **mechanical waves**, where the disturbance involves some kind of *physical displacement* of the medium. Almost all such waves can be classified as being either **transverse** or **longitudinal** waves. A transverse wave causes the medium to displace in a direction *perpendicular* to the direction of the wave motion. A ripple generated on a rope by a sideways flick of the wrist or "The Wave" in a stadium are examples of transverse waves. A longitudinal wave causes the medium to move back and forth *parallel* to the direction that the wave is moving. Sound waves (which are waves of compression and rarification in air) and/or "car waves" in a traffic jam are examples of longitudinal waves. Transverse and longitudinal waves are illustrated in figure E15.2.

Water waves are somewhat peculiar in that as a wave passes, a given "piece" of water actually moves in a small vertical circle around its rest position (see Figure E15.3). Those of you who have played in the surf at a beach know that in front of a wave crest, water moves backwards toward the wave and upward as the crest approaches, but after it has passed the water moves forward with the wave and downward. Thus water waves exhibit *both* longitudinal and transverse motions (though the net displacement of the water after the wave has passed is still zero). Most mechanical waves, though, are either clearly longitudinal or clearly transverse.

Waves carry energy

One of the most important features of mechanical waves is that they carry not only information that a disturbance has occurred but also *energy* away from the disturbance. For example, the water waves moving away

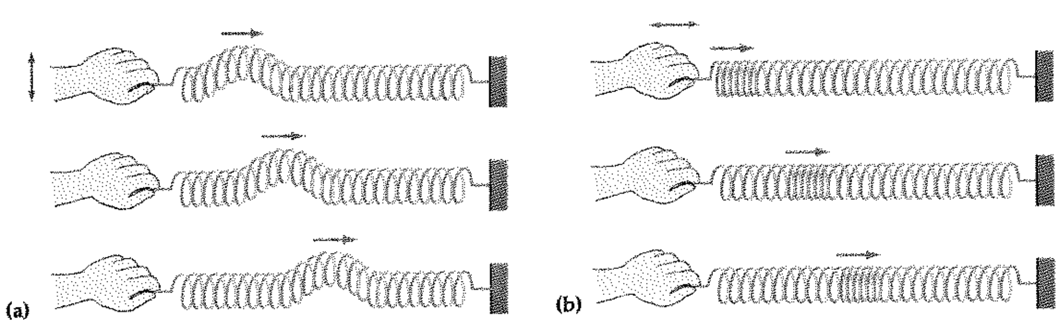


Figure E15.2
(a) A transverse wave moving along a stretched spring. As the wave passes, each element of the spring is displaced perpendicular to the wave's motion. (b) A longitudinal wave on a stretched spring. As the wave passes, each element of the spring is displaced parallel to the wave's motion.

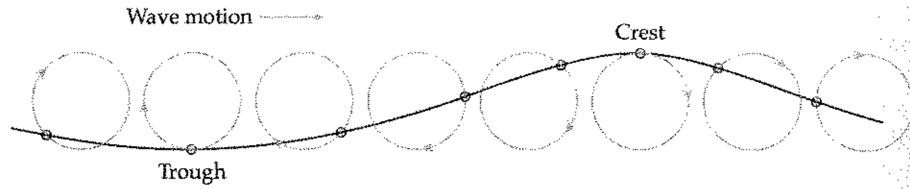


Figure E15.3
This diagram illustrates how particles on the surface of a body of water go around in nearly circular paths as a water wave passes.

from a splash can cause a distant floating bottle to bob up and down as the waves pass; the waves thus transfer energy from the splash and convert it to kinetic energy in the bobbing bottle.

Self-Test E15X.1

An earthquake occurs when part of the earth's crust suddenly slips relative to its surroundings. Such an event radiates energy in the form of two different types of *seismic waves* in the crust of the earth. *P waves* cause the crust to oscillate back and forth toward and away from the earthquake epicenter. *S waves* cause the crust to oscillate up and down. Which of these types is a transverse wave? Which is a longitudinal wave?

Self-Test E15X.2

Describe some evidence that seismic waves carry energy.

E15.2 A Sinusoidal Wave

Why sinusoidal waves are worth studying

A **sinusoidal wave** is a special kind of wave that is especially easy to describe mathematically. Realistic waves are often *approximately* sinusoidal, so a sinusoidal wave represents a convenient simplified model of such waves. But sinusoidal waves are important for another reason. A mathematical theorem called **Fourier's theorem** that states that any wave, no matter how complicated in shape or behavior, can be treated as a superposition of sinusoidal waves. This means that if we fully understand how *sinusoidal* waves behave in a given situation, we essentially understand how *any* wave would behave.

Fourier's theorem: any wave = a sum of sinusoidal waves

Fourier's theorem is an extremely important and useful theorem which you will certainly encounter more than once if you proceed in the study of physics and/or engineering. Its proof, unfortunately, is somewhat beyond our means and would be tangential to our purposes in any case. It is sufficient for our purposes at present for you to understand that not only do sinusoidal waves represent a good approximation to many kinds of real waves, but they actually represent the key to understanding all kinds of waves.

The general mathematical representation of a wave

We can represent *any* wave mathematically by describing a function $f(t,x,y,z)$ that quantifies the disturbance of the medium at every position in space at every instant of time. In this text, I am essentially going to ignore the y and z coordinates and focus on waves that depend on x and t alone: we can pretty much learn everything we need to know about wave behavior from such **one-dimensional** waves without the added complexity of dealing with the y and z coordinates.

A one-dimensional sinusoidal wave has the simple mathematical form

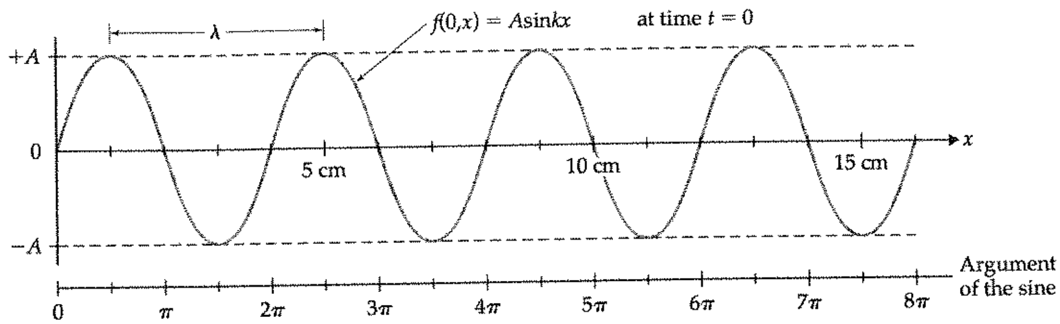


Figure E15.4
A graph of a sinusoidal wave as a function of x at time $t = 0$. In this case, $k = 2\pi/(4\text{ cm}) \approx 1.57\text{ cm}^{-1}$.

$$f(t,x) = A\sin(kx - \omega t) \tag{E15.1}$$

A sinusoidal wave

where $f(t,x)$ quantifies the disturbance of the medium at time t and position x, and A, k, and ω are constants. (Please do not confuse k in this context with the k that we have previously encountered as the Coulomb constant.) What does this sinusoidal wave look like?

A "snapshot" of the wave helps us define its amplitude, wavelength, and wavenumber

We can take a "snapshot" of this wave at time $t = 0$ by setting $t = 0$ in equation E15.1 and drawing a graph of how the disturbance $f(x)$ depends on x at this time. Such a graph is shown in Figure E15.4. Notice how the wave looks like an undulating sequence of hills and valleys (called **crests** and **troughs**). We can see also that the wave disturbance value oscillates between $+A$ and $-A$. The quantity A, which is called the **amplitude** of the wave, thus characterizes the maximum strength of the disturbance.

The distance between two adjacent crests in such a graph is called the **wavelength** λ of the sinusoidal wave This wavelength is related to the constant k as follows. The first crest of the wave to the right of $x = 0$ occurs where $kx_1 = \pi/2$, as you can see from Figure E15.4. The next crest happens when $kx_2 = 5\pi/2 = 2\pi + \pi/2$. The distance between these crests is thus:

$$\lambda \equiv x_2 - x_1 = \frac{1}{k}\left(\frac{5\pi}{2} - \frac{\pi}{2}\right) = \frac{2\pi}{k} \tag{E15.2}$$

You can think of the quantity k as expressing the number of radians-worth of oscillation the wave goes through in a unit distance:

$$k = \frac{2\pi}{\lambda} = \frac{\text{radians/cycle}}{\text{distance/cycle}} = \frac{\text{radians}}{\text{distance}} \tag{E15.3}$$

This quantity is called the **wavenumber** of the wave.

A wave's behavior at a fixed position defines its period, frequency, and angular frequency

Now let us consider what happens to the wave in time as we watch it from a particular *place*, say, $x = 0$. A graph of the sinusoidal wave as a function of time at $x = 0$ is shown in Figure E15.5. Note that we see the wave move up and down between $+A$ and $-A$ as time passes.

The **period** of the wave T is defined to be the time between adjacent crests. By analogy to how we determined the wavelength, you can show that the period is related to ω as follows

$$T = \frac{2\pi}{\omega} \tag{E15.4}$$

Self-Test E15X.3

Verify equation E15.4

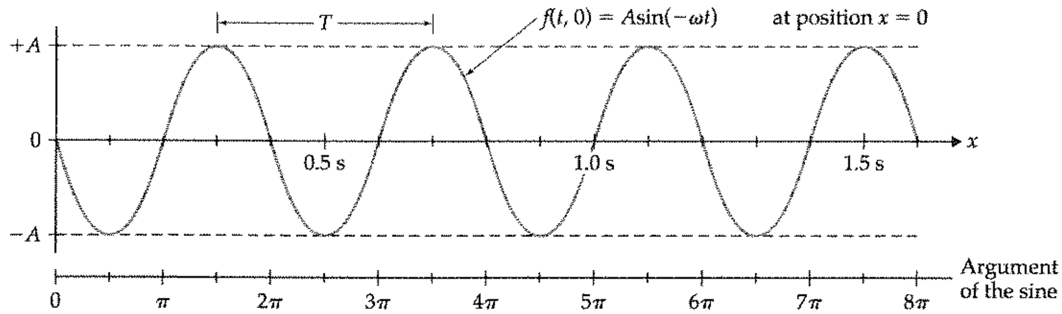


Figure E15.5
A graph of a sinusoidal wave as a function of t at position $x = 0$. In this case, $\omega = 2\pi/(0.4\ \text{s}) \approx$ 15.7 (radians) per second.

The quantity ω can be thought of as expressing the number of radians-worth of oscillation that the wave moves through per unit time:

$$\omega = \frac{2\pi}{T} = \frac{\text{radians/cycle}}{\text{time/cycle}} = \frac{\text{radians}}{\text{time}} \qquad \text{(E15.5)}$$

The constant ω is called the **phase rate** (see chapter N11) or more commonly the **angular frequency** of the oscillation.

The ordinary **frequency** of the oscillation f (in cycles per second, or Hz) is defined to be equal to $1/T$:

$$f = \frac{\text{cycles}}{\text{second}} = \frac{1}{\text{seconds/cycle}} = \frac{1}{T} = \frac{\omega}{2\pi} \qquad \text{(E15.6)}$$

[Note that the f here is not related to the function $f(t,x)$ considered earlier.]

Self-Test E15X.4

If a sinusoidal water wave has a wavelength of 2.0 cm and a frequency of 2.0 Hz, what are the values (with appropriate units) of the waves' wavenumber k and angular frequency ω?

So in summary, here is the constellation of equations that describe a one-dimensional sinusoidal wave:

A summary of the one-dimensional sinusoidal wave formula and associated quantities

$$f(t,x) = A\sin(kx - \omega t) \qquad \text{(E15.7)}$$

$$\text{where} \quad k \equiv \frac{2\pi}{\lambda}, \quad \omega \equiv \frac{2\pi}{T}, \quad f = \frac{\omega}{2\pi} = \frac{1}{T} \qquad \text{(E15.8)}$$

Purpose: These equations describe an idealized one-dimensional sinusoidal wave that varies with time t and position x along the x axis.

Symbols: $f(t,x)$ quantifies the "disturbance" the wave represents at point x at time t, A is the wave's **amplitude**, k (not the Coulomb k!) its **wavenumber**, λ its **wavelength**, ω its **angular frequency**, T its **period**, and f its **frequency** (don't confuse this with the disturbance function).

Limitations: This is an idealization of a real wave.

E15.3 The Phase Velocity of a Wave

Features (such as crests) of a traveling wave move at a rate we call the *phase velocity*

The wave $f(t,x) = A\sin(kx - \omega t)$ has one other important feature: *it moves* as time progresses. Figure E15.6 shows successive snapshots of such a wave at various different times. You can see in this diagram that a given crest of the wave progresses to the right as time passes. We call a wave whose basic spatial shape is translated in space like this as time passes a **traveling wave**: most of the waves we encounter in nature are traveling waves.

Why does the sinusoidal wave given by $f(t,x) = A\sin(kx - \omega t)$ move like this? Consider a given crest of the wave, say, the first crest to the right of $x = 0$

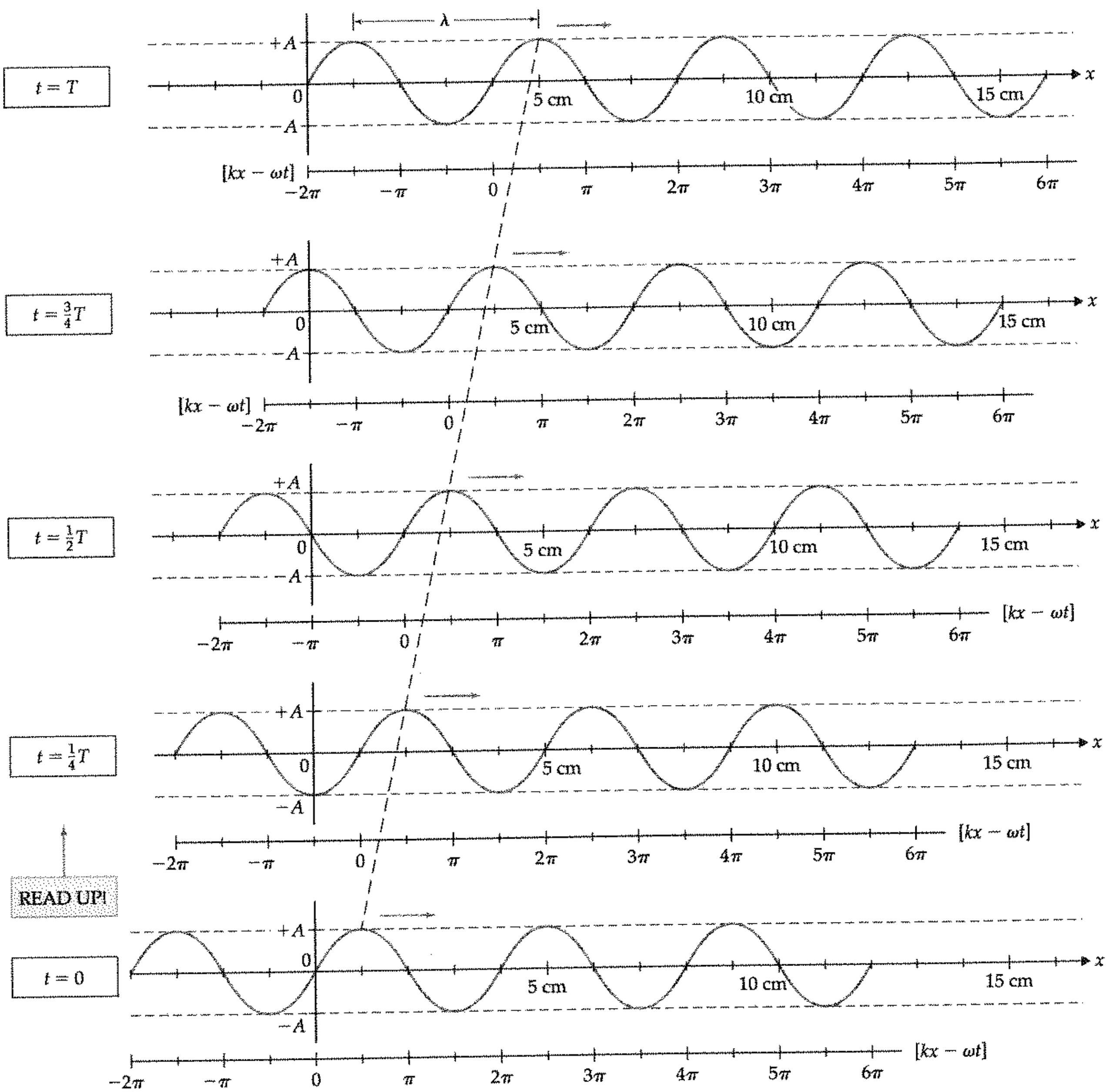


Figure E15.6
A successive series of snapshots of a sinusoidal wave (read from the bottom up!). Note how the crest that was originally at $x = 1$ cm moves to the right as time passes.

at time $t = 0$. This particular crest is the place where the argument of the sine (the quantity in the parentheses that the sine function operates on) has the value $\pi/2$ (the first positive angle where sine becomes +1). So for all time, the location x_{crest} of *this* particular crest is specified by the condition that

$$\frac{\pi}{2} = kx_{\text{crest}} - \omega t \tag{E15.9}$$

Now, at time $t = 0$, this crest is located where $kx_{\text{crest}} = \pi/2$, that is at $x_{\text{crest}} = \pi/2k$. But as t increases, x_{crest} must also increase in proportion to keep the *difference* in equation E15.9 fixed. In fact, if we take the time derivative of both sides of equation E15.9, we find that

$$0 = k\frac{dx_{\text{crest}}}{dt} - \omega \quad \Rightarrow \quad \frac{dx_{\text{crest}}}{dt} = +\frac{\omega}{k} \tag{E15.10}$$

This crest thus moves in the $+x$ direction with speed ω/k.

Self-Test E15X.5

Show that the crest corresponding to the place where the argument of the sine is $5\pi/2$ moves with the same velocity.

Definition of the phase velocity of a sinusoidal wave

The velocity of a given feature (like a given crest) of a traveling wave is called the wave's **phase velocity** (don't confuse this with the wave's *phase rate* w). We see that our sinusoidal traveling wave has a phase velocity in the $+x$ direction whose magnitude (the wave's **phase speed**) is

$$v = |v_x| = +\frac{\omega}{k} \quad \text{(phase speed of our sinusoidal wave)} \tag{E15.11}$$

This equation, in combination with equations E15.8, implies the following relationships between the phase speed and the wave's wavelength, period, and frequency:

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f \tag{E15.12}$$

Purpose: This equation describes how we can calculate a sinusoidal wave's phase speed v from information about its angular velocity ω, its wavenumber k, its wavelength λ, its period T and/or its frequency f.

Limitations: This expression applies only to sinusoidal traveling waves.

We can understand $v = \lambda f$ more intuitively as follows. Consider figure E15.5, and imagine that we sit at the position $x = \pi/2$ and watch the sine wave pass by. At time $t = 0$, there was a crest at this position. After time T has passed, the wave goes through one complete oscillation at our position, so there is again a crest passing our position. Meanwhile, the original crest has moved exactly one wavelength λ ahead in space. The speed of this crest is thus indeed l/T, as claimed by equation E15.12.

Example E15.1 Wavelength from Speed and Frequency

Problem The sound wave from a flute playing the A above middle C has a frequency of 440 Hz. If sound waves move at a speed of 340 m/s in air at 20°C, what is the approximate wavelength of this wave (assuming it is sinusoidal)?

Solution According to equation E15.12, we have:

$$\lambda = \frac{v}{f} = \frac{340\ \text{m}/\not{s}}{440\ \not{\text{Hz}}}\left(\frac{1\ \not{\text{Hz}}}{1\ \text{cycle}/\not{s}}\right) = 0.77\ \text{m} = 77\ \text{cm}. \qquad \text{(E15.13)}$$

Self-Test E15X.6

Seismic *P*-waves radiating from an earthquake travel at a speed of very roughly 6 km/s near the Earth's surface. If such waves for a given earthquake have a period of 0.2 s, what is their wavelength?

E15.4 The Wave Equation

One of the most important equations in physics is the **wave equation**:

The wave equation

$$0 = b\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} \qquad \text{(E15.14)}$$

Purpose: If this equation (where b is a constant independent of t and x) accurately describes the behavior of a disturbance $f(t,x)$ in a medium, that medium will support traveling waves.

Limitations: This equation applies only to cases where the disturbance depends only on one spatial coordinate x.

Note: The partial derivative symbols remind us that when we evaluate the derivative of $f(t,x)$ with respect to one of the variables t or x, we treat the other variable as if it were a constant.

This equation appears again and again in all areas of physics: it accurately describes mechanical disturbances of a stretched string or spring, pressure or density disturbances in solids, liquids, and gases, plasma oscillations in the ionosphere, electrical disturbances in a coaxial cable, , some kinds of quantum-mechanical wave functions, and so on and so on. Physicists rapidly learn to recognize this equation as the basic indicator that traveling sinusoidal disturbance waves are possible in the medium in question.

A proof that our sinusoidal traveling wave is a solution of the wave equation

Let us show that our sinusoidal traveling wave $f(t,x) = A\sin(kx - \omega t)$ is indeed a solution of this equation. If we take the derivative of $f(t,x)$ with respect to x (while treating t as constant), we find that the chain rule tells us that

$$\frac{\partial f}{\partial x} = A\frac{\partial}{\partial x}\sin(kx - \omega t) = A\cos(kx - \omega t)\frac{\partial}{\partial x}(kx - \omega t)$$

$$= A\cos(kx - \omega t)(k) = kA\cos(kx - \omega t) \qquad \text{(E15.15)}$$

If we take the derivative again, we get

$$\frac{\partial^2 f}{\partial x^2} = kA\frac{\partial}{\partial x}\cos(kx-\omega t) = -k^2 A\sin(kx-\omega t) \tag{E15.16}$$

In a similar way, you can show that

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 A\sin(kx-\omega t) \tag{E15.17}$$

Self-Test E15X.7

Verify that equation E15.17 is correct.

Plugging these results into the left side of equation E15.14, we get

$$\begin{aligned} b\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2} &= -b\omega^2 A\sin(kx-\omega t)+k^2 A\sin(kx-\omega t) \\ &= \left(k^2-b\omega^2\right)A\sin(kx-\omega t) \end{aligned} \tag{E15.18}$$

This will satisfy the wave equation as long as $k^2 - b\omega^2 = 0$. We see that a medium obeying the wave equation can indeed support traveling waves as long as the relationship between the values of k and ω for those waves is such that $k^2 = b\omega^2$, where b is whatever constant appearing in the wave equation.

What will be the phase speed of these waves? Equation E5.11 tells us that the phase speed of the wave is $v = \omega/k$. This means that valid sinusoidal traveling-wave solutions to the wave equation will *all* move at the speed

The relationship between the wave speed and the constant b in the wave equation

$$v = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{b}} = \frac{1}{\sqrt{b}} \tag{E15.19}$$

independent of their wavelength or frequency. Therefore, *the value of the constant b appearing in a given medium's wave equation uniquely determines the phase speed of waves moving through that medium.* This is a *very* important conclusion.

One can in fact show (see problem E15S.4) that an arbitrary *sum* of sinusoidal traveling waves also satisfies this equation (as long as $k^2 = b\omega^2$ for each wave in the sum, i.e. each wave moves at speed $v = [b]^{-1/2}$). Since the Fourier theorem tells us that any arbitarily shaped traveling wave can be written as a sum of sinusoidal traveling waves, this means that *any* traveling wave will satisfy the wave equation as long as it moves with phase speed $v = [b]^{-1/2}$.

How the wave equation emerges from basic physics in the context of a taut string

How would we know whether this equation "accurately describes the behavior of a disturbance" in a given medium? Let's see how this works by considering the special case of a transverse wave on a stretched string.

Example E15.2 Waves on a Stretched String

Problem Imagine that we place a string under tension by exerting a tension force of magnitude F_T on its ends. Assume that the string has a mass per unit length of μ. Show that small transverse disturbances on this string obey the wave equation and determine the phase speed v of traveling waves on this string.

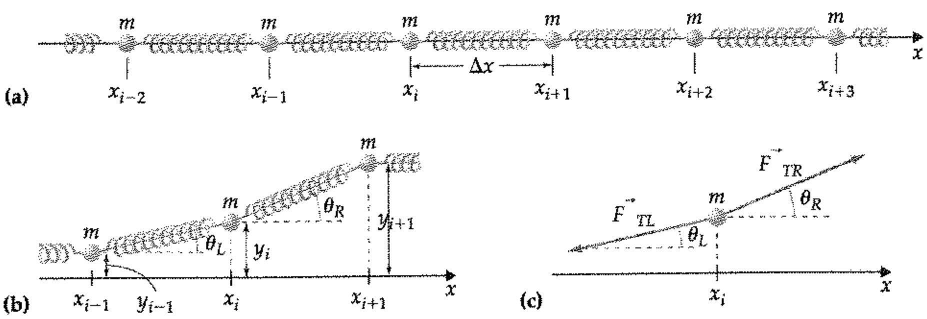


Figure E15.7
(a) We can model a stretched string as being a sequence of particles with mass m connected by springs. The diagram shows the string in its equilibrium state. (b) This diagram shows a possible set of vertically disturbed positions for the ith particle and its nearest neighbors. (c) This diagram displays the forces that are exerted on the ith particle by the springs connecting it to its nearest neighbors.

Translation and Model We will model the string as being a series of particles of mass m connected by identical springs, as shown in figure E15.7a. (We can eventually take the limit that the distance Δx between the masses goes to zero to better model a continuous string.) In our model, saying that the tension on the string has a magnitude of F_T means that each spring is stretched sufficiently so that each of its ends exerts a force of magnitude F_T on the mass to which that end is connected.

Figure E15.7a shows the string in its "undisturbed" configuration, where the string is straight and each mass is at rest on the x axis. We consider the string's ith mass (the one at position x_i) to be "disturbed" if it is displaced vertically away from the x axis to a nonzero y coordinate y_i. A listing of the y coordinates $y_i(t, x_i)$ for all the masses on the string at a given time t completely describes the wave on that string at that time. In this case, therefore, $y_i(t, x_i)$ corresponds to the disturbance function we more generally described earlier as being $f(t, x)$.

How will the masses respond to being disturbed? Figure E15.7b shows some of the masses in a disturbed configuration. The forces acting on the ith mass in this case are the leftward and rightward tension forces $\vec{F}_{TL}$ and $\vec{F}_{TR}$ shown in figure E15.7c. Newton's second law for that mass therefore reads

$$m\vec{a}_i = \vec{F}_{\text{net},i} = \begin{bmatrix} F_{TL,x} + F_{TR,x} \\ F_{TL,y} + F_{TR,y} \\ F_{TL,z} + F_{TR,z} \end{bmatrix} = \begin{bmatrix} -F_{TL}\cos\theta_L + F_{TR}\cos\theta_R \\ -F_{TL}\sin\theta_L + F_{TR}\sin\theta_R \\ 0 \end{bmatrix} \qquad \text{(E15.20)}$$

At this point, I am going to make an approximation. During a realistic oscillation, the angle that any part of the string makes with the horizontal direction is going to be very small (imagine, for example, a vibrating guitar string: it remains almost straight even as it vibrates, right?). Indeed, I have greatly exaggerated the string's curvature in figure E15.7 just to make the angles visible at all. In this "small oscillation" limit, no individual spring will be stretched much more or less than the general stretching that gives the string its tension F_T. Therefore, the magnitude of the force that each individual spring exerts will be essentially equal to F_T. Moreover, in this small oscillation limit, the angles θ_L and θ_R are small. This means that $\cos\theta_L \approx \cos\theta_R \approx 1$, so the x component of the net force on the ith mass is $(F_{\text{net},i})_x = F_{TL}\cos\theta_L - F_{TR}\cos\theta_R \approx F_T - F_T \approx 0$. It is therefore an good approximation in this limit to assume that the x-position of any mass on this string is essentially fixed.

In the small angle limit, we also have $\sin\theta_L \approx \tan\theta_L$ and $\sin\theta_R \approx \tan\theta_R$. If we take the angles to be positive when measured counterclockwise from the x direction, then

$$\sin\theta_L \approx \tan\theta_L = \frac{y_i - y_{i-1}}{\Delta x} = \frac{\Delta y_L}{\Delta x} \tag{E15.21a}$$

$$\sin\theta_R \approx \tan\theta_R = \frac{y_{i+1} - y_i}{\Delta x} = \frac{\Delta y_R}{\Delta x} \tag{E15.21b}$$

where Δx is the horizontal distance between masses, y_i is the vertical position of the ith mass, y_{i-1} is the same for the adjacent mass to the left, y_{i+1} is the same for the adjacent mass to the right, $\Delta y_L \equiv y_i - y_{i-1}$, and $\Delta y_R \equiv y_{i+1} - y_i$. If we plug this back into equation E15.20, we see that the only significant component of the ith mass' acceleration is the y component, whose value is

$$a_{i,y} = \frac{(F_{\text{net},i})_y}{m} \approx \frac{F_T}{m}\left[\sin\theta_R - \sin\theta_L\right] \approx \frac{F_T}{m}\left(\frac{y_{i+1} - y_i}{\Delta x} - \frac{y_i - y_{i-1}}{\Delta x}\right)$$

$$= \frac{F_T \Delta x}{m}\left[\frac{1}{\Delta x}\left(\frac{\Delta y_R}{\Delta x} - \frac{\Delta y_L}{\Delta x}\right)\right] \tag{E15.22}$$

Now, let us look at the quantity in square brackets in this expression in the limit that the spacing Δx between masses becomes very small. In that limit, $y_i(t, x_i)$ will become a continuous function $y(t,x)$. Now, $\Delta y_R / \Delta x$ describes the string's slope for the step just to the right of the ith mass: this ratio best approximates the derivative dy/dx of the continuous function at a point halfway between the ith and (i+1)th mass, i.e., at $x = x_i + \frac{1}{2}\Delta x$. Similarly, $\Delta y_L / \Delta x$ describes the string's slope for the step just left of the ith mass and best approximates dy/dx at $x = x_i - \frac{1}{2}\Delta x$. Therefore, in the limit that $\Delta x \to 0$,

$$\lim_{\Delta x \to 0}\left[\frac{1}{\Delta x}\left(\frac{\Delta y_R}{\Delta x} - \frac{\Delta y_L}{\Delta x}\right)\right] = \lim_{\Delta x \to 0}\left[\frac{(dy/dx)_{x_i+\Delta x/2} - (dy/dx)_{x_i-\Delta x/2}}{\Delta x}\right]$$

$$\equiv \frac{\partial^2 y}{\partial x^2} \quad \text{(evaluated at position } x_i \text{ and time } t\text{)} \tag{E15.23a}$$

since this amounts to the definition of the derivative of the derivative of y with respect to x. By the definition of acceleration, we also have

$$a_{i,y} \equiv \frac{d^2 y_i}{dt^2} = \frac{\partial^2 y}{\partial t^2} \quad \text{(evaluated at position } x_i \text{ and time } t\text{)} \tag{E15.23b}$$

Finally, in this same limit

$$\lim_{\Delta x \to 0}\frac{m}{\Delta x} \equiv \mu \equiv \text{mass per unit length on the string} \tag{E15.24}$$

Solution Plugging these results back into equation E15.22, we therefore have, in the limit that $\Delta x \to 0$,

$$\frac{\mu}{F_T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad \Rightarrow \quad 0 = \frac{\mu}{F_T}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} \tag{E15.25a}$$

This has the same mathematical form as the wave equation [considering that our disturbance function is $y(x,t)$ instead of $f(x,t)$] with $b = F_T / \mu$. We can thus conclude that traveling-wave solutions are possible for transverse disturbances on a stretched string and such waves will move with the phase speed

$$v = \sqrt{\frac{F_T}{\mu}} \tag{E15.25b}$$

Evaluation This result makes good intuitive sense: experience with stretched strings suggest that that the speed of waves on the string would increase if we increase the string tension and/or decrease the string's density.

Self-Test E15X.8

Show that F_T/μ has the units of a squared speed, and calculate the speed of transverse waves on a string whose mass per unit length is 2.0 g/m and whose tension force is 100 N (roughly 22 lbs).

Example E15.2 illustrates how basic physical principles applied to a segment of a stretched string leads directly to the wave equation (in the small-oscillation limit, at least). One finds that the same kind of thing happens in a variety of media (particularly in the small-oscillation limit): this equation is a very common outcome of such analyses! (See the problems for other examples.)

The wave equation is a powerful and useful mathematical tool, but it is somewhat abstract. How can we recognize more intuitively when a medium can support travelning waves? Also, we have seen that sinusoidal traveling waves are a solution to the wave equation, but *must* waves travel in media that obey the wave equation? If so, why? The answers to these questions are linked.

At its most fundamental level, the wave equation links the acceleration $\partial^2 f/\partial t^2$ of a medium's displacement at a point to the curvature $\partial^2 f/\partial x^2$ of the displacement in that point's neighborhood. Useful mnemonics might be that "the tow is equal to the bow" or "the kick is equal to the kink." The point is that a medium whose elements interact in a way that seeks to flatten out any disturbance (attempting to restore the graph of the disturbance to a straight line) qualitatively satisfies the wave equation, and it strictly satisfies the wave equation if the restoring force applied on an element of the medium is exactly proportional to how far it is out of line.

A simple example of such a medium is the torsion-rod wave machine shown in figure E15.8. The medium in this case consists of transverse rods connected to a longitudinal wire spine. The spine in this case strongly twists each rod in direct proportion to the degree to which that rod is out of line with its

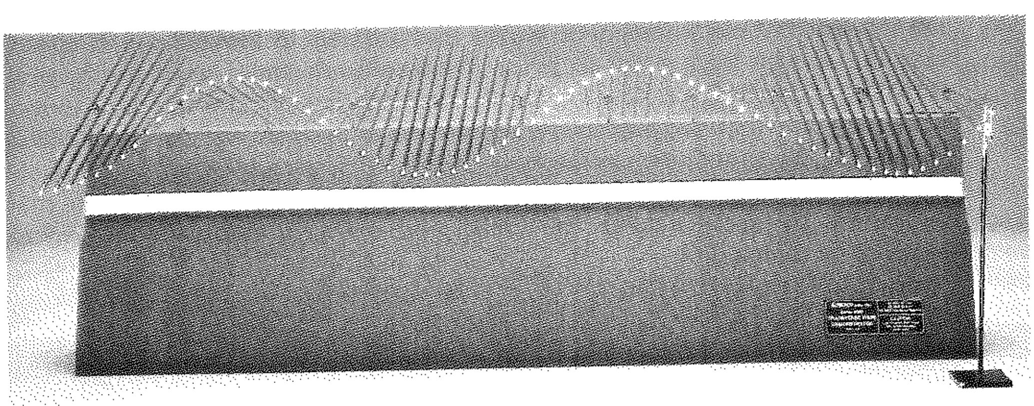


Figure E15.8 A torsion rod wave machine.

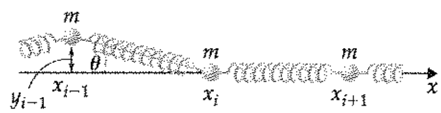


Figure E15.9
Disturbing the mass element at position x_{i-1} creates a kink in the disturbance at position x_i that accelerates the mass element there. But this creates a kink at position x_{i+1} that accelerates the mass element there, and so on.

neighbors, and the spine also twists the rod in a direction that will bring it more in line. Similarly, figure E15.7 on page 281 shows that a mass element on a stretched string only experiences a net force if it is out of the line defined by its nearest neighbors. So a medium obeying the wave equation is easily recognized by its tendency to flatten out any disturbance.

Why do disturbances in such a medium move? Imagine that we are given a stretched string in equilibrium (with all mass elements horizontal and at rest) and we suddenly displace the (i–1)th mass element, as shown in figure E15.9. In that figure, note that this creates a kink in the slope of the displacements in the neighborhood of the ith mass element. By the wave equation, this element will therefore be accelerated upward to try to move this element into the line defined by its neighbors. But its motion upward then creates a kink in the slope at the (i+1)th element, which then accelerates upward, and so on. You can see how this progression causes a disturbance wave to move outward from the initial disturbance.

Moreover, the *speed* at which this disturbance wave moves depends entirely on how rapidly the out-of-line mass element is accelerated: the bigger the acceleration, the more rapidly each mass element will respond to a kink in the disturbance, and the more rapidly the next element will see a kink developing, and so on. Indeed, if we rewrite the wave equation as follows

$$0 = b\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial t^2} = \frac{1}{b}\frac{\partial^2 f}{\partial x^2} \tag{E15.26}$$

you can easily see that the constant of proportionality that expresses how large an acceleration $\partial^2 f/\partial t^2$ is caused by a given disturbance curvature $\partial^2 f/\partial x^2$ is simply $1/b = v^2$. This supports the line of reasoning just given: the stronger the acceleration caused by a given disturbance curvature, the more rapidly disturbance waves will move through the medium.

E15.5 The Mystery du Jour

We are finally in a position to appreciate fully one of the most profound and important consequences of our dynamic electromagnetic field equations: the existence of electromagnetic waves that are able to move through space, carrying energy from point to point. We now understand what mechanical traveling waves are like, how to describe them mathematically, and how to recognize media that support traveling waves. But what *are* electromagnetic waves? In what medium would such travel? What corresponds to "displacements" of that medium? What are the physical effects that seek to restore that medium to flatness? These are the questions that we will resolve in the crowning chapter of this unit.

HOMEWORK PROBLEMS

Basic Skills

E15B.1 Sound waves move through air at a speed of about 340 m/s. Compute the wavelength of the following sound waves:
(a) an organ pipe playing middle C (260 Hz)
(b) the highest audible pitch (≈ 20,000 Hz)
(c) the lowest audible pitch (≈15 Hz)

E15B.2 In the next chapter, we will find out that electromagnetic waves move at the speed of light. What are the wavelengths of the following kinds of electromagnetic waves?
(a) radio waves on the AM band (≈ 1000 kHz)
(b) radio waves on the FM band (≈ 100 MHz)
(c) EM waves in a microwave oven (≈ 30 GHz)

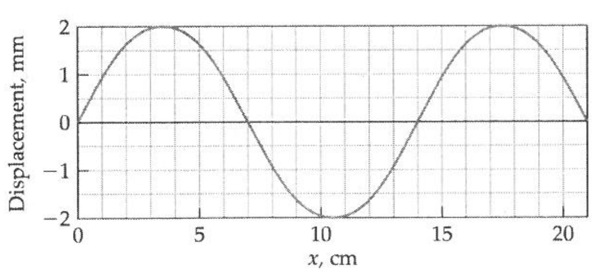
E15B.3 Sound waves move through air at a speed of about 340 m/s. What would be the frequency of a sound wave that has a wavelength of 1 m? 1 inch? 1 mm?

E15B.4 Visible light has wavelengths between 700 nm and about 400 nm. If light really is an electromagnetic wave, then what are the corresponding frequencies of these waves?

E15B.5 A sinusoidal traveling water wave has a wavelength of 25 cm and a frequency of 0.60 Hz. What are k and ω for this wave? What is the phase speed of this wave?

E15B.6 A sinusoidal wave moving down a taut rope has a wavelength of 2.0 m and a period of about 0.5 s. What are k and w for this wave? What is the phase speed of the wave?

E15B.7 Consider the sinusoidal traveling wave shown below (this is a snapshot at a certain instant of time). Assume the wave travels at 1.0 m/s.
(a) What is the wave's amplitude?
(b) What is its wavenumber k?
(c) What is its angular velocity ω?
(d) What is its period?
(e) What is its frequency?



Synthetic

E15S.1 Sinusoidal water waves are created 120 km offshore by an earthquake near a small island. Observers in helicopters above the island report that the waves have an amplitude of about 2.0 m, a wavelength of 15 m, and a frequency of about 0.5 Hz. How long do lifeguards on the mainland have to evacuate beaches before the waves arrive?

E15S.2 magine that a geologist is measuring the waves produced by small earthquakes using two seismographs, one 12 km from the volcano, and another 17 km from the volcano. During one earthquake, the waves feel like the gentle rocking of a boat at a frequency of about 1.5 Hz and an amplitude of about 1 cm. The geologist later notices that the closer seismograph registered the waves about 0.85 s sooner than the other. What was the approximate wavelength of the waves during this episode?

E15S.3 Consider the function $f(x,t) = A\sin(kx+\omega t)$. Does this function describe a *traveling* sinusoidal wave? If not, why not? If so, what is the speed (in terms of w and k) and the direction of motion of this wave? Does this wave satisfy the wave equation? Explain your responses carefully.

E15S.4 Argue that if $f(t,x)$ and $g(t,x)$ separately satisfy the wave equation for a given medium, then $h(t,x) = f(t,x) + g(t,x)$ also satisfies the wave equation. (By extension, any sum of sinusoidal waves will satisfy the wave equation.)

E15S.5 By rocking a boat, a person produces water waves on a previously undisturbed lake. This person observes that the boat oscillates 12 times in 20 s, each oscillation producing a wave crest 5 cm above the undisturbed level of the lake, and that the waves reach the shore (12 m away) in about 6 s. At any given instant of time, about how many wave crests are there between the boat and the shore?

E15S.6 Consider a series of identical masses m arranged along the x axis that are connected by identical springs with spring-constant k_s. The masses are all free to slide in the $\pm x$ direction on a frictionless surface. Assume that when all the masses are in their equilibrium positions, their centers are equal distances Δx apart and the springs between them are all relaxed. Let's define the position x_i of the ith mass (when all masses are in equilibrium) to be its "home" position. We can then define the "disturbance" s_i of the ith mass at a given time to be its horizontal displacement from its home position, where s_i is positive if the mass is displaced in the $+x$ direction from home, and negative if it is displaced in the $-x$ direction from home.

(a) Make a careful drawing of the mass at a given arbitrary position x and its two adjacent neighbors and argue that the x component of the net force on the mass at a given instant of time is

$$F_x = k_s \Delta s_R - k_s \Delta s_L \qquad \text{(E15.29)}$$

where $\Delta s_R \equiv s_{i+1} - s_i$ describes how much larger the distance between the mass at x_i and the mass to the right is than the usual separation Δx, and $\Delta s_L \equiv s_i - s_{i-1}$ is the same for distance to the left of that mass. (A positive value of Δs_R or Δs_L means that the spring between the masses is stretched; a negative value means that it is compressed.)

(b) Argue (using the definition of the double derivative) that if Δx is reasonably small,

$$\frac{1}{\Delta x^2}[\Delta s_R - \Delta s_L] \approx \frac{\partial^2 s}{\partial x^2} \qquad \text{(E15.30)}$$

where in the last step, we are imagining $s(x,t)$ to be a smooth function that matches the value of s_i at each home position x_i.

(c) Use this to argue that

$$F_x \approx k_s \Delta x^2 \frac{\partial^2 s}{\partial x^2} \qquad \text{(E15.31)}$$

(d) Show that Newton's second law and the previous results together imply that longtitudinal disturbances in this set of interconnected masses obey the wave equation

$$0 = b\frac{\partial^2 s}{\partial t^2} - \frac{\partial^2 s}{\partial x^2} \qquad \text{(E15.32)}$$

and find the wave speed v in terms of k_s, m, and Δx. (This means that disturbances in this system will move like traveling waves up and down the x axis. If we consider the masses to be atoms and the springs to be interatomic bonds, this could represent a simplified model of a one-dimensional elemental solid.)

E15S.7 Figure E15.8 on page 303 shows a torsional wave machine of a type commonly used for classroom demonstrations of traveling waves. The wave machine consists of rods of length L and mass m separated by distance Δx along a wire spine. The disturbance function in this case is the angle $\theta(t,x)$ that the rod at position x makes with the horizontal plane at time t. When a segment of the spine of length Δx is twisted through a small angle $\Delta\theta$, the segment exerts a torque on the rod at each end whose magnitude is

$$\tau = k_t \frac{\Delta\theta}{\Delta x} \qquad \text{(E15.33)}$$

where k_t is a constant expressing the spine's stiffness (a kind of a spring constant for twisting). Assume that the x axis is along the spine and the positive direction of that axis is toward the right.

(a) Argue that the x component of the net torque on the rod at x_i is

$$\tau_{\text{net},x} = k_t(\Delta\theta_R - \Delta\theta_L) \qquad \text{(E15.34)}$$

where $\Delta\theta_R = \theta(x_{i+1}) - \theta(x_i)$ is the angle that the rod to the right of x_i is twisted relative to the angle of the rod at x_i, and $\Delta\theta_L = \theta(x_i) - \theta(x_{i-1})$ is the angle that the rod at x_i is twisted relative to the next rod to the left. (Conventionally, counterclockwise angles are positive and clockwise angles are negative.)

(b) Argue (using the definition of the double derivative) that if the distance Δx between adjacent rods is reasonably small,

$$\frac{1}{\Delta x^2}[\Delta\theta_R - \Delta\theta_L] \approx \frac{\partial^2\theta}{\partial x^2} \qquad \text{(E15.35)}$$

(c) As we saw in unit C, the definition of torque is $\vec{\tau}_{\text{net}} \equiv d\vec{L}/dt$, $\vec{L} = \frac{1}{12}ML^2\vec{\omega}$ for a long, thin rod of mass m and length L, and $\omega \equiv d\theta/dt$. Use this information to argue that the rightward component of the torque on the rod is related to its angle according to the expression

$$\tau_{\text{net},x} \propto \frac{\partial^2\theta}{\partial t^2} \qquad \text{(E15.36)}$$

and find the constant of proportionality.

(d) Link equation E15.34 and E15.36 to show that an angular disturbance on this medium obeys the wave equation and to determine the speed of disturbance waves in terms of m, L, and k_t

E15S.8 Consider a series of identical disk-shaped magnets strung along like beads along a thin rod. The magnets are able to slide frictionlessly along the rod, and are oriented so that each magnet repels both its nearest neighbors. Assume that when all the magnets are at rest at their equilibrium positions, their centers are equal distances Δx apart. Let's define the position x_i of the ith magnet (when all the magnets are in this equilibrium state) to be its "home" position,. We can then define the "disturbance" s_i for the ith magnet at a given time to be its horizontal displacement from its home position, where s_i is positive if the magnet is displaced in the $+x$ direction from home, and negative if it is displaced in the $-x$ direction. The magnets all have mass m, and

ANSWERS TO SELF-TESTS

E15X.1 *P*-waves are longitudinal, *S*-waves transverse.

E15X.2 Earthquake waves shake objects, which means that the waves must have given the objects kinetic energy. This energy ultimately comes from the sudden relaxation of strains in rock due to the slippage along a fault at the epicenter. This energy may be carried many miles from the epicenter by the wave.

E15X.3 The first crest to pass $x = 0$ after $t = 0$ is when $-\omega t_1 = -3\pi/2$ [since $\sin(-3\pi/2) = +1$]. The next crest passes when $-\omega t_2 = -7\pi/2$. Thus $T \equiv t_2 - t_1 = -2\pi/(-\omega) = 2\pi/\omega$, as claimed.

E15X.4 $k = 3.14/\text{cm}$ and $\omega = 12.57/\text{s}$

E15X.5 (The calculation is essentially identical with what we did before except that we now substitute $5\pi/2$ everywhere that we had $\pi/2$ before.)

E15X.6 Solving equation E15.12 for λ and plugging in the numbers, we get $\lambda = 1.2$ km

E15X.7 The first derivative of $f = A\sin(kx - \omega t)$ is

$$\frac{\partial f}{\partial t} = A\frac{\partial}{\partial t}\sin(kx-\omega t) = A\cos(kx-\omega t)\frac{\partial}{\partial t}(kx-\omega t) = -\omega A\cos(kx-\omega t) \qquad \text{(E15.43)}$$

Taking the derivative again, we get

$$\frac{\partial^2 f}{\partial t^2} = -\omega A\frac{\partial}{\partial t}\cos(kx-\omega t) = +\omega A\sin(kx-\omega t)\frac{\partial}{\partial t}(kx-\omega t) = -\omega^2 A\sin(kx-\omega t) \qquad \text{(E15.44)}$$

E15X.8 We can do both parts at once here:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{100\,\cancel{\text{N}}}{2.0\,\cancel{\text{g}}/\text{m}}\left(\frac{1\,\cancel{\text{kg}}\cdot\text{m/s}^2}{1\,\cancel{\text{N}}}\right)\left(\frac{1000\,\cancel{\text{g}}}{1\,\cancel{\text{kg}}}\right)} = \sqrt{50{,}000\,\frac{\text{m}^2}{\text{s}^2}} = 220\text{ m/s} \qquad \text{(E15.45)}$$

Note that the units do work out correctly!

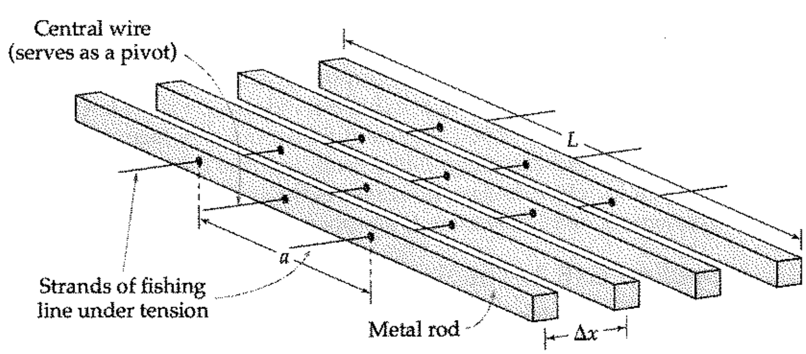


Figure E15.10
A schematic diagram indicating the construction of a certain wave machine. The square metal rods can rotate about the central wire, but the two stretched fishing lines on either side of the central wire try to keep the rods level. How fast do waves move on this wave machine? (See problem E15R.3.)